Vacuum state of the quantum string without anomalies in any number of dimensions *

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Abstract

We show that the anomalies of the Virasoro algebra are due to the asymmetric behavior of raising and lowering operators with respect to the ground state of the string. With the adoption of a symmetric vacuum we obtain a non-anomalous theory in any number of dimensions. In particular for D=4.

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1 Introduction

String theory has been a source of many excellent works on the subject. Some of them give rise to the hope that a "unified and final" description of all forces of nature can be achieved.

However the appearance of anomalies seems to impose a severe restriction on the number of dimensions of the spaces in which such a program can be succesfully carried out.

In reference [1] a didactic exposition of string theory can be found. We will adopt this book as our reference text on the subject. See also references [2] and [3].

The position of the string in space time is given by the coordinates $X^\mu(\sigma, \tau)$. The action can be written:

$$S = -\frac{1}{2\pi} \int d^2\sigma \;\; \eta^{\alpha\beta} \;\; \partial_\alpha X \cdot \partial_\beta X \tag{1}$$

where $\eta^{\alpha\beta}$ is a two dimensional Minkowsky metric tensor.

The action (1) leads to the wave equation

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0 \tag{2}$$

Its general solution is a superposition of a right moving coordinate X_R^μ and a left moving one X_L^μ . For the closed string [4] :

$$X_R^\mu = \frac{1}{2} x^\mu + \frac{1}{2} p^\mu (\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)} \tag{3}$$

$$X_L^\mu = \frac{1}{2} x^\mu + \frac{1}{2} p^\mu (\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)} \tag{4}$$

For the open string we have standing waves:

$$X^\mu = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \tag{5}$$

The Virasoro generators can be defined as the Fourier components of $\dot{X}^2$ [5]:

$$L_m = \frac{1}{2\pi}\int_0^{\pi} d\sigma \;\; e^{-2im\sigma}\dot{X}_R^2 = \frac{1}{2}\sum_{-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \tag{6}$$

$$\tilde{L}_m = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{-2im\sigma}\dot{X}_L^2 = \frac{1}{2}\sum_{-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \tag{7}$$

(with $\alpha_0^\mu = p^\mu$).

2 Quantization

With the string action given by (1), the momentum canonically conjugate to the coordinate X^μ is proportional to $\dot{X}^\mu$. Using (3) and (4) it is possible to find the commutation relations obeyed by the coefficients α_m^μ [6] :

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \tag{8}$$

Except for normalization, eq. (8) is the usual rule for the harmonic oscilator raising and lowering operators.

The commutation relations obeyed by the Virasoro operators can be found by using (6) and the rules given by (8). The result is the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} \;\;,\;\; (m+n \neq 0) \tag{9}$$

When n=-m, one must be careful. A c-number central term may appear and there are some ordering ambiguities.

We define:

$$[L_m, L_n] = (m-n)L_{m+n} + A_m\delta_{m+n} \tag{10}$$

Where A_m is called the anomaly of the Virasoro algebra. If we take n=-m in (10) we have:

$$[L_m, L_{-m}] = 2mL_0 + A_m \tag{11}$$

In view of eq.(6), the natural form for L_0 is:

$$L_0 = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \alpha_n\cdot\alpha_{-n}) \tag{12}$$

However, a normal ordered expression for L_0 is generally understood. Anyway, a c-number addition to L_0 is equivalent to a redefinition of the anomaly A_m.

If L_0 is defined so as to have a null vacuum expectation value, then

$$A_m = \langle 0 \mid [L_m, L_{-m}] \mid 0 \rangle \tag{13}$$

3 The vacuum state

To build-up the states of the string, it is necessary to complement the basic commutation rules (eq.(8)), with a definition of the vacuum.

The usual definition takes the vacuum as the state annihilated by all α_m^μ with $m > 0$. As we are going to change this identification for the string, some words are needed to support such an attitude.

There are good reasons to adopt the costumary definition. In particle physics it leads to a positive definite spectrum for the hamiltonian. It also leads to the Feynman propagator, which satisfies the asymptotic requirement that the positive energies propagate towards the future and the negative energies propagate backwards in time.

However, for waves in a string (either open or closed), there are no asymptotic requirements in space. These waves are confined to the string and can not be observed as free waves. An analogous example is provided by waves that obey a Klein-Gordon equation with complex-mass parameter [7]. It is not possible to see those waves in free states, as they blow-up asymptotically. Something similar happens with waves associated with the possible existence of tachyons [8] [9]. See also ref [10].

In all those cases the vacuum plays a symmetrical role with respect to the raising and lowering operators.

It is also possible that the ground state has been incorrectly identified, as mentioned in ref. [11]. In which case the symmetrical vacuum seems to be a reasonable alternative to the usual definition.

These considerations lead us to the assumption that the vacuum state of the string should be annihilated not by α_m^μ (or α_{-m}^μ), but rather by the symmetrized product $\{\alpha_m^\mu, \alpha_{-m}^\mu\}$ (no summation over μ or m).

In other words, the vacuum obeys (See refs. [7] to [10])

$$(\alpha^\mu_m \alpha^\mu_{-m} + \alpha^\mu_{-m}\alpha^\mu_m) \mid 0\rangle = 0 \quad (no\ summation) \tag{14}$$

As a consequence we have

$$L_0 \mid 0\rangle = 0 \tag{15}$$

where L_0 is defined by eq.(12).
It also follows that, for any μ, ν, m, n:

$$\langle 0 \mid \alpha^\mu_m \alpha^\nu_n + \alpha^\nu_n \alpha^\mu_m \mid 0\rangle = 0 \tag{16}$$

As

$$\alpha^\mu_m \alpha^\nu_n = \frac{1}{2}\left[\alpha^\mu_m, \alpha^\nu_n\right] + \frac{1}{2}\{\alpha^\mu_m, \alpha^\nu_n\}$$

it is easy to see that :

$$\langle 0 \mid \alpha^\mu_m \alpha^\nu_n \mid 0\rangle = \frac{1}{2} m \delta_{m+n} \eta^{\mu\nu} \tag{17}$$

while in the usual case:

$$\begin{aligned} \langle 0 \mid \alpha^\mu_m \alpha^\nu_n \mid 0\rangle &= m\delta_{m+n}\eta^{\mu\nu} \quad if \;\; m > 0 \\ &= 0 \quad if \;\; m < 0 \end{aligned} \tag{18}$$

We would like to point out that when the vacuum state is not anihilated by the decreasing operator, a set of negative normed states appears.
For example, if we define

$$\mid -1_n\rangle \equiv \alpha_n \mid 0\rangle \quad n\rangle 0$$

Then, according to eq.(17):

$$\langle -1_n \mid -1_n\rangle = \langle 0 \mid \alpha^+_n \alpha_n \mid 0\rangle = -\frac{1}{2} n$$

And the states $\mid -1_n\rangle$ (n¿0) are negatively normed.
However, as explained in references [7] to [10] , we have to take into account that different choices of the vacuum imply different propagators. For

the usual case one obtains Feynman's Green function while for the symmetrical vacuum we get Wheeler's propagator (half advanced and half retarded [12]). As is well-known, Feynman's causal function has an on-shell pole, which signals the existence of a corresponding free state. On the other hand, Wheeler's propagator as an on-shell **zero** . Thus implying that the corresponding free mode can not be excited. So that here the negative normed states are harmless.

It is to be noted that, in spite of its advanced component, the Wheeler's Green function does not give rise to causal inconsistencies [13] .

4 The anomalies

Before engaging in an actual calculation, it seems convenient to give an intuitive approach to the subject. Let us first consider the usual case. L_0 is supposed to be normal ordered, so that eq.(13) is valid. For $m > 0$ we have:

$$\alpha_m^\mu \mid 0\rangle = 0 \quad and \quad L_m \mid 0\rangle = 0 \tag{19}$$

On the other hand,

$$L_{-m} \mid 0\rangle = \frac{1}{2}\sum_{n=1}^{m-1} \alpha_{n-m}\cdot\alpha_{-n} \mid 0\rangle \tag{20}$$

So that:

$$A_m = \langle 0 \mid [L_m, L_{-m}] \mid 0\rangle = \langle 0 \mid L_m L_{-m} \mid 0\rangle$$

$$A_m = \frac{1}{4}\sum_{n=1}^{m-1}\sum_{s=1}^{m-1}\langle 0 \mid \alpha_{m-n}\cdot\alpha_n \; \alpha_{s-m}\cdot\alpha_{-s} \mid 0\rangle$$

$$= \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\sum_{n=1}^{m-1}\sum_{s=1}^{m-1}\langle 0 \mid \Big(\alpha^\mu_{m-n}\left[\alpha^\nu_n, \alpha^\rho_{s-m}\right]\alpha^\sigma_{-s} +$$

$$\alpha^\mu_{m-n}\alpha^\rho_{s-m}\alpha^\nu_n\alpha^\sigma_{-s}\Big) \mid 0\rangle = \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\sum_{n=1}^{m-1}\sum_{s=1}^{m-1}$$

$$\left(\left[\alpha^\mu_{m-n}, \alpha^\sigma_{-s}\right]\left[\alpha^\nu_n, \alpha^\rho_{s-m}\right] + \left[\alpha^\nu_{m-n}, \alpha^\rho_{s-m}\right]\left[\alpha^\nu_n, \alpha^\sigma_{-s}\right]\right)$$

$$A_m = \frac{D}{2}\sum_{n=1}^{m-1} n(n-m) = \frac{D}{2}\frac{m(m^2-1)}{6} \tag{21}$$

Suppose now, for the sake of the argument, that we define the vacuum to be annihilated by α^{μ}_{-m} with $m > 0$ (Time inverted case) :

$$\alpha^{\mu}_{-m} \mid 0\rangle = 0 \quad , \quad L_{-m} \mid 0\rangle = 0 \tag{22}$$

The corresponding anomaly would be :

$$A^{'}_m = \langle 0 \mid [L_m, L_{-m}] \mid 0\rangle = -\langle 0 \mid L_{-m}L_m \mid 0\rangle$$

$$A^{'}_m = -\frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\sum_{n=1}^{m-1}\sum_{s=1}^{m-1}\left([\alpha^{\mu}_{s-m}, \alpha^{\sigma}_{n}]\right.$$

$$\left.\left[\alpha^{\nu}_{-s}, \alpha^{\rho}_{m-n}\right] + [\alpha^{\mu}_{s-m}, \alpha^{\rho}_{m-n}]\left[\alpha^{\nu}_{-s}, \alpha^{\sigma}_{n}\right]\right)$$

$$A^{'}_m = -\frac{D}{2}\frac{m(m^2-1)}{6} \tag{23}$$

A mere change of sign with respect to (21).

One can see the influence of the vacuum state (either (19) or (22)) on the value of the anomaly (resp.(21) or (23)). Thus,it is understandable that the symmetrical identification expressed by eq.(14), should lead to the disappearance of the anomaly.

To check the last assertion we are going to consider again eq.(13) but this time together with:

$$\langle 0 \mid \alpha^{\mu}_{m}\alpha^{\nu}_{n} \mid 0\rangle = m\varepsilon_m\delta_{m+n}\eta^{\mu\nu} \tag{24}$$

Eq.(24) covers all three possibilities, namely:

I: The usual case,

$$\begin{aligned} \varepsilon^{I}_{m} = \quad & 1 \quad if \quad m > 0 \\ & 0 \quad if \quad m < 0 \end{aligned} \tag{25}$$

II: The inverted case,

$$\begin{aligned} \varepsilon^{II}_{m} = \quad & 0 \quad if \quad m > 0 \\ & 1 \quad if \quad m < 0 \end{aligned} \tag{26}$$

III: The symmetrical case,

$$\varepsilon_m^{III} = \frac{1}{2} \quad (any\ m) \tag{27}$$

Let us now take a generic term in (13) (We will not write explicitly the space-time indices):

$$\langle 0 \mid \alpha_{m-n}\alpha_n\alpha_{-m-s}\alpha_s \mid 0\rangle = [\alpha_n, \alpha_{-m-s}]\cdot$$

$$\cdot\langle 0 \mid \alpha_{m-n}\alpha_s \mid 0\rangle + \langle 0 \mid \alpha_{m-n}\alpha_{-m-s}\alpha_n\alpha_s \mid 0\rangle =$$

$$n\delta_{n-m-s}\varepsilon_{m-n}(m-n) + n\delta_{n+s}\varepsilon_{m-n}(m-n)+$$

$$\langle 0 \mid \alpha_{m-n}\alpha_{-m-s}\alpha_s\alpha_n \mid 0\rangle = n(m-n)\varepsilon_{m-n}\delta_{n-m-s}+$$

$$n(m-n)\varepsilon_{m-n}\delta_{n+s}\delta_{-n-s} + (m-n)s\varepsilon_s\delta_{n+s}+$$

$$(m-n)(-m-s)\varepsilon_{-m-s}\delta_{n-m-s} + \langle 0 \mid \alpha_{-m-s}\alpha_s\alpha_{m-n}\alpha_n \mid 0\rangle$$

$$\langle 0 \mid [\alpha_{m-n}\alpha_n, \alpha_{-m-s}\alpha_s] \mid 0\rangle = n(m-n)\left(\varepsilon_{m-n} - \varepsilon_{-n}\right)\left(\delta_{n-m-s} + \delta_{n+s}\right) \tag{28}$$

The last factor tells us that there is a possible contribution only when s=-n or s=n-m. Let us examine the ε -factor. For case I (eq.(25)), and $m > 0$, there is no contribution for n outside the interval $1 \leq n \leq m-1$. With this information it is easy to reproduce eq.(21).

For case II (eq.(26)) and $m > 0$, the contributing interval is the same. But in that interval we have $\varepsilon_{m-n}^{II} = 0$, while in case I we had $\varepsilon_{-n}^{I} = 0$. This implies a change of sign with respect to (21) as in eq.(23).

Finally, for case III (eq.(27)), $\varepsilon_m^{III} = \frac{1}{2}$, so that the ε -parenthesis is zero and no anomaly is present.

A similar calculation can be carried out for the "Lorentz anomaly" [14]. Its value depends on Δ_m defined in ref. [14], are all null for the symmetrical vacuum.

5 The ghosts

It is possible to follow a path-integral method for the treatment of string motions. L.D.Faddeev and N.Popov procedure [15] [16] to handle the gauge-fixing determinants leads to the use of anti-commuting ghost fields, which should be considered together with normal string states.

The ghost action in the conformal gauge, can be written [17]

$$S^c = \frac{1}{\pi}\int d^2\sigma \left(c^+\partial_- b_+ + c^-\partial_+ b_-\right) \tag{29}$$

where the ghost and anti-ghost fields obey the anticommutation relations

$$\{b(\sigma,\tau), c(\sigma^{'},\tau)\} = 2\pi\delta(\sigma-\sigma^{'}) \tag{30}$$

We can now transform to normal mode coordinates.

$$c^{\pm} = \sum_{-\infty}^{\infty} c_n e^{-in(\tau\pm\sigma)} \tag{31}$$

$$b_{\pm} = \sum_{-\infty}^{\infty} b_n e^{-in(\tau\pm\sigma)} \tag{32}$$

where c_n and b_n obey (from (30)):

$$\{c_m, b_n\} = \delta_{m+n} \tag{33}$$

The Fourier components of the world-sheet energy-momentum tensor define the Virasoso generators [17]:

$$L^c_m = \sum_{-\infty}^{\infty}(m-n)b_{m+n}c_{-n} \tag{34}$$

They obey a Virasoro algebra with a possible central term:

$$[L^c_m, L^c_n] = (m-n)L^c_{m+n} + A^c_n\delta_{m+n} \tag{35}$$

For the evaluation of A^c_m we will follow a method similar to the one used in § 4.

We define three different vacuum states:

$$I \quad b_n \mid 0\rangle = 0 \ \ , \ \ c_n \mid 0\rangle = 0 \ \ , \ \ n<0 \tag{36}$$

$$II \quad b_{-n} \mid 0\rangle = 0 \ \ , \ \ c_{-n} \mid 0\rangle = 0 \ \ , \ \ n < 0 \tag{37}$$

$$III \quad (b_n c_{-n} - c_{-n} b_n) \mid 0\rangle = 0 \ \ , \ \ any\ n \tag{38}$$

The symmetrical ground state case III, has also been used to define the vacuum corresponding to fermion fields with complex-mass parameter [19].

The contribution to the anomaly, produced by the ghosts, can now be determined for the three cases.

L_0^c is normal ordered so that:

$$A_m^c = \langle 0 \mid \left[L_m^c, L_{-m}^c \right] \mid 0\rangle \tag{39}$$

Also

$$\langle 0 \mid b_m c_n \mid 0\rangle = \varepsilon_n \delta_{n+m} \tag{40}$$

Where

$$\begin{aligned} \varepsilon_n^I = \ & 1 \ , \ if \ \ n > 0 \\ & 0 \ , \ if \ \ n < 0 \end{aligned} \tag{41}$$

$$\begin{aligned} \varepsilon_n^{II} = \ & 0 \ , \ if \ \ n > 0 \\ & 1 \ , \ if \ \ n < 0 \end{aligned} \tag{42}$$

$$\varepsilon_n^{III} = \frac{1}{2} \ \ , \ \ any\ n \tag{43}$$

Now we take a generic term of the product $L_m^c L_n^c$:

$$b_{m+r} c_{-r} b_{n+s} c_{-s} = b_{m+r} \{c_{-r}, b_{n+s}\} c_{-s} -$$

$$b_{m+r} b_{n+s} c_{-r} c_{-s} = \delta_{n+s-r} b_{m+r} c_{-s} -$$

$$b_{n+s} \{b_{m+r}, c_{-s}\} c_{-r} + b_{n+s} c_{-s} b_{m+r} c_{-r}$$

Leading to:

$$\langle 0 \mid [b_{m+r} c_{-r}, b_{n+s} c_{-s}] \mid 0\rangle = \delta_{m+n} \delta_{m+r-s} \left(\varepsilon_{-s} - \varepsilon_{-r}\right)$$

i.e.:

$$
A_m^c = \langle 0 \mid \left[L_m^c, L_{-m}^c \right] \mid 0 \rangle =
$$

$$
\sum_s (m-s)(2m+s)\left(\varepsilon_{-m-s} - \varepsilon_{-s}\right) \tag{44}
$$

Again, we can see from (44) that the anomaly changes sign from I to II, and disappears in case III.

6 Discussion

In this paper we wanted to emphasize that the anomalies are consequences of the asymmetry of the ground state with respect to the lowering (α) and raising (α^+) operators. As a result, it follows that the choice of the vacuum for the string determines the anomaly of the Virasoro algebra. When the symmetry is restored, the anomaly disappears.

As a matter of facts, we face three typical situations. First (and most important), by succesive applications of α we arrive at a state $\mid 0\rangle$ such that $\alpha \mid 0\rangle = 0$. Second, the inverted case where by succesive applications of α^+ we arrive at a "ceiling state" $\alpha^+ \mid 0\rangle = 0$. Third, no matter how many times we multiply by α or α^+, we never annihilate a state. Those are the cases examined in § 3 and § 4. They lead to different matrix representations of α and α^+ . Also they imply, of course, different propagators for the fields that obey the oscillator differential equation. The first case corresponds to the usual Klein-Gordon equation and the associated Feynman propagator. The second case takes place if we adopt the time inverted solution and the complex-conjugate of the Feynman propagator. The third alternative occurs when the field can not be observed asymptotically (propagating freely). Such is the case for example, when the Klein-Gordon equation has a complex-mass parameter and the free solution blows-up asymptotically. The associated propagator is half advanced and half retarded (See ref. [7]).

In the latter case, as the negative normed states can not be excited, they can be considered to be auxiliary constructions that may latter be discarded. Only the propagators are needed for the subsequent developments of the theory.

Sometimes doubts are raised about the costumary identification of the vacuum (See ref. [11]). Also, the asymptotic requirements leading to the

Feynman propagator do not seem to be compelling for the finite string. For all those facts we think that a reasonable alternative to the usual identification is provided by the symmetrical vacuum state. Furthermore, in this way we obtain a non-anomalous theory in any number of dimensions, in particular in D=4. Consequently, we think that the possibilities opened up by the adoption of the symmetrical vacuum are worth exploring.

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